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Quantum revivals and carpets in some exactly solvable systems

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Abstract. We consider the revival properties of quantum systems with an eigenspectrum $E_n \propto n^2$, and compare them with the simplest member of this class—the infinite square well. In addition to having perfect revivals at integer multiples of the revival time t_R, these systems all enjoy perfect fractional revivals at quarterly intervals of t_R. A closer examination of the quantum evolution is performed for the Pöschel–Teller and Rosen–Morse potentials, and comparison is made with the infinite square well using quantum carpets.

Over the past ten years or so there has been a growing interest in the quantum dynamics of simple systems, motivated in part by the richness of new phenomena such as revivals [1] and quantum carpets [2–4]. Indeed, the phenomenon of revivals is not merely a theoretical construct, but has been observed in ion traps [5], Rydberg atoms [6] and semiconductor wells [7] and has been used to differentiate ionization pathways in potassium dimers [8]. Quantum revivals are similar, but distinct from, quantum Poincaré recurrences, which have been studied recently in the context of the kicked rotator [9]. In the former, one is interested in the deterministic reconstruction of the wavefunction during its evolution inside a fixed potential, whilst in the latter, interest is focused on the decay of the return probability in 'mixed' regions of phase space, as a measure of the quantum chaos in the (usually forced) system. For the most part, analytic studies of revivals and carpets have concentrated on the infinite square well (ISW) potential, which is known to have perfect revivals *and* fractional revivals [10], and also a quantum carpet composed of rays (straight lines in the space–time plane) [3]. These properties have been understood on the basis of the quadratic dependence of the energy on quantum number, along with the fact that the eigenfunctions are elementary trigonometric functions.

It is well understood that perfect revivals can only occur for systems whose energy spectrum is purely quadratic in the quantum number [1]. If the dependence is purely linear (harmonic oscillator) the only timescale is the classical period of oscillation, while for more complicated energy spectra, revivals will be imperfect due to modulations from the super-revival timescale. Although many studies have been devoted to the simplest quantum system with a quadratic energy spectrum—namely, the ISW—there has been less attention paid to the host of other potentials which share this property (although see [11] for a discussion of the autocorrelation function for the Morse potential). It is guaranteed that these systems will have perfect revivals, but what can one say about fractional revivals, and the existence of quantum carpets (i.e. hidden structures in the space–time plot of the probability density)?

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We shall begin with some very general remarks about fractional revivals. Consider a system with a purely quadratic, nondegenerate energy spectrum $E_n = \alpha^2 n^2$, $(n = 0, 1, 2, \ldots)$, and with a potential $V(x)$ centred at $x = 0$. We take the potential to be an *even* function of x. In this case the eigenfunctions $\phi_n(x)$ will have a definite even or odd symmetry, alternating as the quantum number increases, the ground state naturally being even, since it has no node. So we have $\phi_n(-x) = (-1)^n \phi_n(x)$. We prepare the wavefunction of the system in an initial state specified by the energy eigenfunction expansion

$$\psi(x, 0) = \sum_n c_n \phi_n(x). \tag{1}$$

We restrict ourselves to contributions from bound states only. The time-evolved wavefunction is given by

$$\psi(x, t) = \sum_n c_n \phi_n(x) \exp[-\mathrm{i} E_n t] \tag{2}$$

where we have chosen units of $\hbar = 1$. Given the quadratic dependence of the energy levels on n, it is easy to see from equation (2) that the wavefunction will be identical to its initial state at integer multiples of the revival time $t_R \equiv 2\pi/\alpha^2$.

Now consider the wavefunction at a time equal to one-half of t_R. One easily finds

$$\psi(x, t_R/2) = \sum_n c_n \phi_n(x) \exp[-\mathrm{i}\pi n^2]. \tag{3}$$

Given that $\mathrm{e}^{-\mathrm{i}\pi n^2} = (-1)^n$, we have

$$\psi(x, t_R/2) = \sum_{n \text{ even}} c_n \phi_n(x) - \sum_{n \text{ odd}} c_n \phi_n(x). \tag{4}$$

Returning to the initial wavefunction, one may use the parity properties of the eigenstates to demonstrate that

$$\begin{aligned} \psi(x, 0) &= \sum_{n \text{ even}} c_n \phi_n(x) + \sum_{n \text{ odd}} c_n \phi_n(x) \\ \psi(-x, 0) &= \sum_{n \text{ even}} c_n \phi_n(x) - \sum_{n \text{ odd}} c_n \phi_n(x). \end{aligned} \tag{5}$$

Clearly, on comparing equations (4) and (5) we have the perfect fractional revival $\psi(x, t_R/2) = \psi(-x, 0)$. This result may appear to follow from the symmetry of the potential and time-reversal invariance; however, this is not the case (cf the discussion following equation (14)).

A less obvious result follows, however, when we study the wavefunction at one-quarter of the revival time. We have

$$\psi(x, t_R/4) = \sum_n c_n \phi_n(x) \exp[-\mathrm{i}\pi n^2/2]. \tag{6}$$

Considering the phase for $n = 0, 1, 2, 3 \bmod (4)$ one can easily establish that

$$\psi(x, t_R/4) = \sum_{n \text{ even}} c_n \phi_n(x) - \mathrm{i} \sum_{n \text{ odd}} c_n \phi_n(x). \tag{7}$$

Solving the two expressions in equation (5) for the odd and even sets of modes, we find the perfect fractional revival

$$\psi(x, t_R/4) = \frac{(1-\mathrm{i})}{2} \psi(x, 0) + \frac{(1+\mathrm{i})}{2} \psi(-x, 0). \tag{8}$$

In a similar manner one may show that

$$\psi(x, 3t_R/4) = \frac{(1+\mathrm{i})}{2} \psi(x, 0) + \frac{(1-\mathrm{i})}{2} \psi(-x, 0). \tag{9}$$

Thus, a system with an even potential and a purely quadratic energy spectrum supports perfect fractional revivals at quarters of the revival time. Especially interesting are the fractional revivals at $t_R/4$ and $3t_R/4$ which for an initially localized wavefunction will consist of two perfect, mirrored 'cat states'. We have failed to find perfect fractional revivals at other fractions of the revival time for this general class of systems (i.e. utilizing only parity properties of the eigenfunctions).

Let us now be more specific, and consider in turn two potentials of the type considered above: namely symmetric cases of the Pöschel–Teller (PT) and Rosen–Morse (RM) potentials [12, 13]. PT takes the form (with the ground state energy set at zero)

$$V_{S1}(x) = -A^2 + A(A-\alpha)\sec^2(\alpha x) \tag{10}$$

defined in the range $-\pi/2 \leqslant \alpha x \leqslant \pi/2$, and with energy spectrum

$$E_n = (A + n\alpha)^2 - A^2. \tag{11}$$

In order to have perfect quarterly revivals, it is necessary to choose $A = M\alpha$, with M an integer. Note that PT has an infinite number of bound states, and no scattering states. The bound states may be expressed in terms of Gegenbauer polynomials with argument $\sin(\alpha x)$. We shall restrict our attention to $M = 2$, in which case the energy eigenfunctions are sums of bilinear products of elementary trigonometric functions.

RM takes the form

$$V_{S2}(x) = A^2 - A(A+\alpha)\,\mathrm{sech}^2(\alpha x) \tag{12}$$

defined for x on the entire real line, and with energy spectrum

$$E_n = A^2 - (A - n\alpha)^2. \tag{13}$$

Again, to ensure perfect quarterly revivals, we choose $A = M\alpha$, with M a positive integer. RM has only M bound states, which may be expressed in terms of the Gegenbauer polynomials with argument $\mathrm{i}\sinh(\alpha x)$.

Although the energy spectra for these potentials are not purely quadratic in n, it is a simple matter to redefine the quantum number by shifting by M, in which case the perfect quarterly revivals found above take the slightly modified form

$$\begin{aligned}
\psi(x, t_R/4) &= \tfrac{1}{2}(1 - \mathrm{i}\theta)\psi(x, 0) + \tfrac{1}{2}(1 + \mathrm{i}\theta)\psi(-x, 0) \\
\psi(x, t_R/2) &= \psi(-x, 0) \\
\psi(x, 3t_R/4) &= \tfrac{1}{2}(1 + \mathrm{i}\theta)\psi(x, 0) + \tfrac{1}{2}(1 - \mathrm{i}\theta)\psi(-x, 0)
\end{aligned} \tag{14}$$

where $\theta = (-1)^M$. If one chooses A/α to be a half-integer, the revival at $t_R/2$ is *identical* to the full revival, yet the perfect quarterly revivals are lost.

Much can be learnt about these systems by preparing an initial wavefunction from a finite number of eigenstates, and then studying its evolution using equation (2). It is common practice [1] to construct the initial wavefunction using the first N modes, with weights c_n drawn from a Gaussian distribution centred at some reasonably energetic mode $\bar{n}$

$$|c_n|^2 \sim \exp\left[-\frac{(n-\bar{n})^2}{2\sigma^2}\right]. \tag{15}$$

This simulates, for instance, a laser-prepared state in an ion trap. We have evolved the wavefunction in the PT and RM potentials using these weights. (We choose equal phases for the c_n in order to create a well-localized initial wavepacket. Choosing random phases does not affect the revival structure, but tends to obscure the regularity of the quantum carpets.) Aside from confirming the perfect quarterly revivals found above, we have found that PT has nearly perfect fractional revival states at rational fractions of the revival time, whereas there

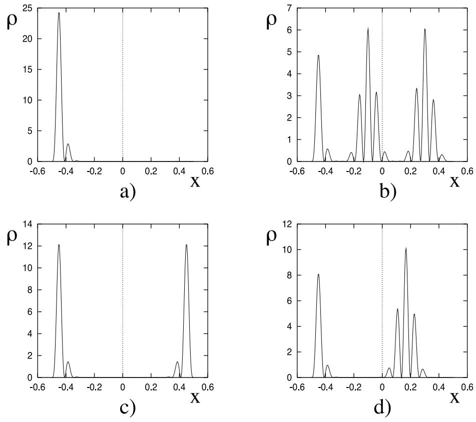

Figure 1. Probability density for the PT potential (with parameter values $\alpha = \pi$, $M = 2$, $N = 30$, $\sigma = 3.0$, and $\bar{n} = 15$) at times: (*a*) $t = 0$, (*b*) $t = t_R/5$, (*c*) $t = t_R/4$, and (*d*) $t = t_R/3$.

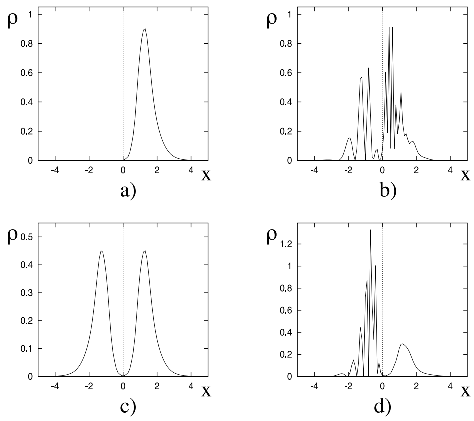

Figure 2. Probability density for the RM potential (with parameter values $\alpha = 1.0$, $M = N = 20$, $\sigma = 4.0$, and $\bar{n} = 10$) at times: (*a*) $t = 0$, (*b*) $t = t_R/5$, (*c*) $t = t_R/4$, and (*d*) $t = t_R/3$.

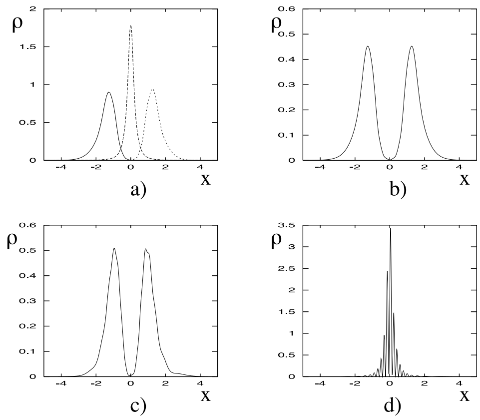

Figure 3. Probability density for the RM potential (with parameter values $\alpha = 1.0$, $M = N = 20$, $\sigma = 4.0$, and $\bar{n} = 10$) for values of $A/\alpha = M + r$, with (*a*) $r = 0.0$ (leftmost), 0.25 (centre), and 0.5 (rightmost) at time $t_R/2$; (*b*) $r = 0.0$ at $t_R/4$; (*c*) $r = 0.25$ at $t = t_R/4$; and (*d*) $r = 0.5$ at $t = t_R/4$.

is little sign of such nearly perfect states for RM (although they will appear for much higher energy wavepackets [1]). In figures 1 and 2 we illustrate this by showing the probability density ρ at times $t = 0, t_R/5, t_R/4$ and $t_R/3$ for PT and RM potentials, respectively. To test the robustness of the quarterly revivals we have also evolved an excited wavepacket in the RM potential, but away from the rationality condition $A/\alpha = M$. We set $A/\alpha = M + r$, with $r \in (0, \frac{1}{2})$. We find imperfect, yet 'smooth' fractional revivals at $t_R/2$ for general values of r (see figure 3(*a*)). However, the fractional revivals at $t_R/4$ are more sensitive to r, and fade away for $r > 0.25$ (see figures 3(*b*)–3(*d*)).

The almost perfect fractional revivals for PT may be understood intuitively, since for a wavefunction constructed around a moderately energetic state, the PT potential closely resembles the ISW—i.e. the harmonic structure of the bottom of the well is barely resolved. (One may also make a more quantitative argument by changing the basis of equation (2) from the PT eigenfunctions to the eigenfunctions of the ISW. One finds that the overlap integrals for large n are sharply peaked. Thus the dominant part of the wavefunction may be expanded (with the weights c_n) in terms of the energy eigenstates of the ISW, which, as mentioned before, has perfect fractional revivals at all rational fractions of t_R.) Given the poor resolution

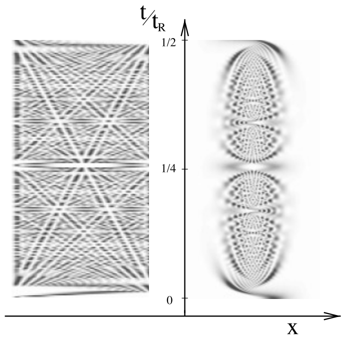

Figure 4. The quantum carpets (i.e. space–time contour plots of the probability density) for PT (left) and RM (right), with parameter values as before, shown for $t \in (0, t_R/2)$. The darker regions indicate higher probability density.

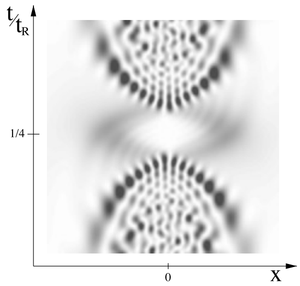

Figure 5. Detail of the quantum carpet for the RM potential around $t = t_R/4$, showing the emergence and subsequent collapse of two perfect cat states.

of the structure of the potential, one would also expect that the quantum carpets for PT closely resemble those found for the ISW. This is indeed the case, as shown on the left of figure 4, where the characteristic rays are clearly visible.

As an alternative to preparing the initial wavefunction around some energetic state, one can use a weighting that favours the lower-lying states (using an exponentially decaying distribution for the c_n, for example). We have studied this case, and indeed the correspondence with the ISW disappears, and the evolution of the wavefunction is a slowly modulated classical oscillation (since the well has a harmonic minimum). The almost perfect fractional revivals are invisible, and the quantum carpet has no structure. The perfect quarterly revivals may still be resolved.

As mentioned above, the RM potential shows little sign of fractional revivals, apart from the perfect quarterly revivals. However, the quantum carpet for this potential reveals considerable structure, as shown on the right of figure 4; the rays of PT are replaced by a complicated structure of what appear to be nonlinear 'world lines'. A magnified view of the first perfect quarterly revival is shown in figure 5.

We have yet to attain a physical understanding of the quantum carpet for RM. Approaches which work well for the ISW [3] are less useful here due to the complicated nature of the eigenfunctions. It is interesting to note that the structures visible in the ISW/PT quantum carpet (i.e. rays) can be interpreted as a superposition of coherent wavepackets (CWPs) which follow classical trajectories (with a discrete spectrum of initial velocities selected to ensure a perfect revival at $t = t_R$). Whether these CWPs can be identified with the coherent states [14] corresponding to the ISW is unclear. The apparent world lines of the RM quantum carpet might also be interpreted in this way, although the CWPs no longer follow classical trajectories. This is clear from the manner in which the world lines proceed through the minimum of the potential at $t = t_R/4$.

A computational application of our results is the testing of numerical algorithms designed to integrate forward the time-dependent Schrödinger equation [15]. These algorithms do not generally rely upon energy eigenfunction expansions, and testing them against exact results (for general initial conditions, and non-trivial potentials) is difficult due to the scarcity of such results in quantum dynamics. An algorithm which integrates the wavefunction forward in time in the PT or RM potentials, and successfully generates (i.e. recovers with good precision) perfect quarterly revivals can be trusted in other applications. A positive feature of this test is that one can implement it for *any* initial condition (strictly true only for PT for which the bound states form a complete set). It is interesting to note that algorithms which use discrete Fourier modes for free-particle propagation will fail to capture perfect revivals, since they are based on a Hamiltonian with a discrete lattice Laplacian, thus replacing the pure k^2 spectrum by $2(1 - \cos k)$, although they will find increasingly good revival structures as the number of Fourier modes is increased.

In conclusion, we have studied the quantum revivals and carpets for systems with a quadratic energy spectrum and an even potential. We have proven that *all* such systems have perfect quarterly revivals, in addition to perfect complete revivals. We have studied the time evolution of two such systems—the PT and RM potentials—more closely. From a moderately energetic initial distribution of modes, the evolution of the wavefunction in PT (being defined in a finite region of space) has many similarities to that of the ISW, with almost perfect fractional revivals at rational fractions of t_R, and a quantum carpet with characteristic rays. This similarity disappears continuously as one decreases the mean energy of the wavefunction, thus allowing better resolution of the harmonic minimum of the well. The evolution of the wavefunction in RM shows little fractional revival structure, apart from the perfect quarterly revivals. Its quantum carpet, although devoid of rays, displays a dazzling pattern, the understanding of which is currently being pursued.

This study has shown that many of the rich dynamical properties of the ISW are fairly generic, thus increasing their experimental relevance. Indeed, the PT and RM potentials capture features of real quantum systems, which are missing in the ISW: namely, a spatially varying potential energy, with a harmonic minimum, and in the case of RM, a finite number of bound states (see also the discussion of super revivals in the finite square well [16]). The robustness of the perfect quarterly revivals may well be of interest to experimentalists seeking to create perfect cat states from a localized wavepacket. Indeed, this is the initial entangled state required for quantum communication [17] (although 'entanglement' usually refers to two or more degrees of freedom), and which is generally created using more complicated laser interferometry. (Fabrication of an approximate RM potential may well be realizable using semiconductor quantum well technology.)

Aside from the PT and RM potentials, there are other potentials which are isospectral to the ISW (and which may be generated using the Darboux transformation [13]). A study of their revival properties may well prove worthwhile. As a final remark, it is noteworthy that the PT potential (with $M = 2$) and the ISW are supersymmetric partner potentials [13], and therefore share the same energy spectrum (bar the lowest state). Whether, due to supersymmetry, these systems share other *dynamical* equivalents, aside from perfect quarterly revivals, is an interesting open question.

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